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# SOME RELATIONS CONNECTING THE SUMS OF THE COAXIAL MINORS OF A CIRCULANT.

By W. H. METZLER.

In 1878 Dr. J. W. S. Glashier called attention\* to the fact that the determinant of order  $n$  formed from subtracting  $x$  from the elements along the principal diagonal of a circulant has as factors :

$$- (x - s) \prod_{k=1}^m \{x^2 - (a_1 + a_2\theta^k + a_3\theta^{2k} + \dots + a_n\theta^{(n-1)k}) \\ \times (a_1 + a_2\theta^{n-k} + a_3\theta^{n-2k} + \dots + a_n\theta^{n-(n-1)k})\}$$

for  $n = 2m + 1$ ,  
and

$$(x - s)(x - s') \prod_{k=1}^{m-2} \{x^2 - (a_1 + a_2\theta^k + a_3\theta^{2k} + \dots + a_n\theta^{(n-1)k}) \\ \times (a_1 + a_2\theta^{n-k} + a_3\theta^{n-2k} + \dots + a_n\theta^{n-(n-1)k})\}$$

for  $n = 2m$ ,  
where

$$s = a_1 + a_2 + a_3 + \dots \\ s' = a_1 - a_2 + a_3 - \dots$$

and  $\theta$  is an imaginary  $n$ th root of unity.

Thus for  $n = 5$  we have

$$\begin{vmatrix} a_1 - x & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 - x & a_4 & a_5 & a_1 \\ a_3 & a_4 & a_5 - x & a_1 & a_2 \\ a_4 & a_5 & a_1 & a_2 - x & a_3 \\ a_5 & a_1 & a_2 & a_3 & a_4 - x \end{vmatrix} \\ = - \{x - (a_1 + a_2 + a_3 + a_4 + a_5)\} \{x^2 - (a_1 + a_2\theta + a_3\theta^2 \\ + a_4\theta^3 + a_5\theta^4)(a_1 + a_2\theta^4 + a_3\theta^3 + a_4\theta^2 + a_5\theta)\} \\ \times \{x^2 - (a_1 + a_2\theta^2 + a_3\theta^4 + a_4\theta + a_5\theta^3)(a_1 + a_2\theta^3 \\ + a_3\theta + a_4\theta^4 + a_5\theta^2)\}.$$

\* *Quarterly Journal of Mathematics*, Vol. XV, pp. 347-356. Cf. Muir, *Messenger of Mathematics*, New Series, No. 491, 1912.

If we represent the imaginary factor of the circulant by  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ , the right-hand side may be written

$$(x-s)(x^2-\alpha_1\alpha_4)(x^2-\alpha_2\alpha_3) \text{ or } (x-s)(x^2-\alpha_{1,4})(x^2-\alpha_{2,3}),$$

where

$$\alpha_{i,j} = \alpha_i \cdot \alpha_j.$$

Expanding both sides in terms of powers of  $x$ , we have in the general case

for

$$n = 2m,$$

$$\begin{aligned} & x^{2m} - x^{2m-1}\Sigma a_{11} + x^{2m-2}\Sigma \binom{12}{12} - \dots - x \cdot \Sigma A_{11} + \Delta \\ & = x^{2m} - x^{2m-1}(s+s') + x^{2m-2}(s \cdot s' - \Sigma \alpha_{1,n-1}) + \dots \\ (A) \quad & + (-1)^{k-1} x^{2m-2k} (s \cdot s' \Sigma \alpha_{1,n-1} \cdot \alpha_{2,n-2} \dots \alpha_{k-1,n-k+1} \\ & - \Sigma \alpha_{1,n-1} \cdot \alpha_{2,n-2} \dots \alpha_{k,n-k}) \\ & + (-1)^{k-1} x^{2m-2k-1} (s+s') \Sigma \alpha_{1,n-1} \cdot \alpha_{2,n-2} \dots \alpha_{k,n-k} + \dots \\ & + (-1)^{m-1} s \cdot s' \alpha_{1,n-1} \cdot \alpha_{2,n-2} \dots \alpha_{m-1,n-m+1}; \end{aligned}$$

for

$$n = 2m+1,$$

$$\begin{aligned} & -x^n + x^{n-1}\Sigma a_{11} - x^{n-2}\Sigma \binom{12}{12} + \dots - x \cdot \Sigma A_{11} + \Delta \\ & = -x^n + x^{n-1}s + x^{n-2}\Sigma \alpha_{1,n} - \dots \\ (B) \quad & + (-1)^{k-1} x^{n-2k} \cdot \Sigma \alpha_{1,n-1} \cdot \alpha_{2,n-2} \dots \alpha_{k,n-k} \\ & + (-1)^{n-1} x^{n-2k-1} \cdot s \cdot \Sigma \alpha_{1,n-1} \cdot \alpha_{2,n-2} \dots \alpha_{k,n-k} + \dots \\ & + (-1)^m \cdot s \cdot \alpha_{1,n-1} \cdot \alpha_{2,n-2} \dots \alpha_{m,n-m}; \end{aligned}$$

Where  $\Sigma a_{11}$  represents the sum of the elements along the principal diagonal of the circulant  $\Delta$ ,  $\Sigma \binom{12 \dots k}{12 \dots k}$ , the sum of the coaxial minors of order  $k$ , and  $\Sigma A_{11}$  the sum of the coaxial minors of order  $n-1$ .

Equating coefficients of like powers of  $x$  we have from (A)

$$\begin{aligned} (s+s')\Sigma \alpha_{1,n-1} \dots \alpha_{k-1,n-k+1} &= (-1)^{k-1} \Sigma \binom{12 \dots 2k-1}{12 \dots 2k-1}, \\ s \cdot s' \Sigma \alpha_{1,n-1} \dots \alpha_{k-1,n-k+1} - \Sigma \alpha_{1,n-1} \dots \alpha_{k,n-k} &= (-1)^{k-1} \Sigma \binom{12 \dots 2k}{12 \dots 2k}, \\ (s+s')\Sigma \alpha_{1,n-1} \dots \alpha_{k,n-k} &= (-1)^k \Sigma \binom{12 \dots 2k+1}{12 \dots 2k+1}, \end{aligned}$$

that is,

$$(-1)^{k-1} s \cdot s' \frac{\Sigma \binom{12 \dots 2k-1}{12 \dots 2k-1}}{s+s'} - (-1)^k \frac{\Sigma \binom{12 \dots 2k+1}{12 \dots 2k+1}}{s+s'} = (-1)^{k-1} \Sigma \binom{12 \dots 2k}{12 \dots 2k}.$$

or

$$s \cdot s' = \frac{(s + s') \Sigma \binom{12 \dots 2k}{12 \dots 2k} - \Sigma \binom{12 \dots 2k+1}{12 \dots 2k+1}}{\Sigma \binom{12 \dots 2k-1}{12 \dots 2k-1}}, \quad (1)$$

a constant ratio for all values of  $k$  from 1 to  $m-1$ .

When  $k=m$ , the ratio becomes

$$\frac{\Sigma a_{11} \cdot \Delta}{\Sigma A_{11}};$$

from B,

$$\Sigma \alpha_{1, n-1} \cdots \alpha_{k, n-k} = (-1)^k \Sigma \binom{12 \dots 2k}{12 \dots 2k},$$

$$s \cdot \Sigma \alpha_{1, n-1} \cdots \alpha_{k, n-k} = (-1)^k \Sigma \binom{12 \dots 2k+1}{12 \dots 2k+1},$$

That is

$$s \cdot \Sigma \binom{12 \dots 2k}{12 \dots 2k} = \Sigma \binom{12 \dots 2k+1}{12 \dots 2k+1}$$

or

$$s = \frac{\Sigma \binom{12 \dots 2k+1}{12 \dots 2k+1}}{\Sigma \binom{12 \dots 2k}{12 \dots 2k}}. \quad (2)$$

In the case where  $n=2m$ , we have

$$s' = \frac{\Sigma \binom{12 \dots 2k+1}{12 \dots 2k+1}}{\Sigma \binom{12 \dots 2k}{12 \dots 2k}}, \quad \text{if } s = 0, \quad (1a)$$

It is known\* that when  $s=0$  all the primary minors of  $\Delta$  are equal. It follows, therefore, that the unique primary minor

$$A_1 = \frac{s' \cdot \Sigma \binom{12 \dots 2m-2}{12 \dots 2m-2}}{2m}. \quad (1b)$$

Similarly

$$s = \frac{\Sigma \binom{12 \dots 2k+1}{12 \dots 2k+1}}{\Sigma \binom{12 \dots 2k}{12 \dots 2k}} \quad \text{when } s' = 0. \quad (1c)$$

If  $s + s' = 0$ , then

$$s^2 = s'^2 = \frac{\Sigma \binom{12 \dots 2k+1}{12 \dots 2k+1}}{\Sigma \binom{12 \dots 2k-1}{12 \dots 2k-1}}. \quad (1d)$$

\* Borchardt, "Ueber eine der Interpolation entsprechende Darstellung der Eliminations-Resultante," *Crelle's Journal*, lvii, pp. 111-121. Cf. Muir, *Messenger of Mathematics*, New Series, No. 536, Vol. XLV, December, 1915.

If  $s = s' = 0$ , then

$$\Sigma_{(12 \dots 2k+1)}^{(12 \dots 2k+1)} = 0. \quad (1e)$$

From (1b) we see that, when  $s = s' = 0$ , the unique minor  $A_1$  vanishes. In this case the determinantal equation contains only even powers of  $x$  and since all its roots are real it follows that the signs must be alternately positive and negative. That is, the signs of  $\Sigma_{(12)}^{(12)}$ ,  $\Sigma_{(1224)}^{(1224)} \dots$  must be alternately negative and positive.

In the case of (1d) we see that the sums of the coaxial minors of odd order have the same sign.

In the case when  $n = 2m + 1$  we have

$$\Sigma_{(12 \dots 2k+1)}^{(12 \dots 2k+1)} = 0 \quad \text{of} \quad s = 0. \quad (2a)$$

Here again the signs of the sums of coaxial minors of even order must be alternately negative and positive.

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